

# Maximal monotonicity, conjugation and the duality product

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## Abstract

Recently, the authors studied the connection between each maximal monotone operator  $T$  and a family  $\mathcal{H}(T)$  of convex functions. Each member of this family characterizes the operator and satisfies two particular inequalities.

The aim of this paper is to establish the converse of the latter fact. Namely, that every convex function satisfying those two particular inequalities is associated to a unique maximal monotone operator.

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## 1 Introduction

Let  $X$  be a real Banach space and  $X^*$  be the dual of  $X$ .

Denote by  $\langle \cdot, \cdot \rangle$  the duality product. A multivalued operator  $T : X \rightrightarrows X^*$  is *monotone* if

$$\langle x^1 - x^2, v^1 - v^2 \rangle \geq 0, \quad \forall v^1 \in T(x^1), v^2 \in T(x^2).$$

Such an operator is *maximal monotone* if its *graph*, that is, the set

$$G(T) = \{(x, v) \in X \times X^* \mid v \in T(x)\},$$

is not properly contained in the graph of any other monotone operator  $T' : X \rightrightarrows X^*$ . We will identify  $T$  with its graph  $G(T)$ .

Rockafellar [6] proved that subdifferentials of proper closed convex functions on  $X$  are maximal monotone. In general maximal monotone operators are not subdifferentials. Even though, maximal monotone operators can be represented by convex functions in  $X \times X^*$ . This is a result of Fitzpatrick [4], recently rediscovered in [5, 2]. Next we describe Fitzpatrick's results. Given a maximal monotone operator  $T : X \rightrightarrows X^*$ , define

$$\mathcal{H}(T) := \left\{ h : X \times X^* \rightarrow \overline{\mathbb{R}} \left| \begin{array}{l} h \text{ convex, closed;} \\ \forall (x, v) \in X \times X^*, h(x, v) \geq \langle x, v \rangle, \\ (x, v) \in T \Rightarrow h(x, v) = \langle x, v \rangle. \end{array} \right. \right\}. \quad (1) \quad \boxed{\text{eq: defh}}$$

Define also  $\varphi_T : X \times X^* \rightarrow \overline{\mathbb{R}}$ ,

$$\varphi_T(x, v) := \sup\{\langle x - y, u - v \rangle \mid (y, u) \in T\} + \langle x, v \rangle.$$

**th:f1** **Theorem 1.1 (Fitzpatrick[4])** *Let  $T : X \rightrightarrows X^*$  be maximal monotone. The function  $\varphi_T$  belongs to  $\mathcal{H}(T)$  and is the smallest function of this family. Moreover, for any  $h \in \mathcal{H}(T)$ ,*

$$(x, v) \in T \Leftrightarrow h(x, v) = \langle x, v \rangle.$$

From the above equivalence, it follows that each  $h \in \mathcal{H}(T)$  fully characterizes  $T$ . Given a function  $f : X \rightarrow \overline{\mathbb{R}}$ , the *Legendre transform* or *conjugate* of  $f$  is defined as  $f^* : X^* \rightarrow \overline{\mathbb{R}}$ ,

$$f^*(v) = \sup\{\langle x, v \rangle - f(x) \mid x \in X\}.$$

Conjugation is an essential tool in the study of convex functions. For  $h : X \times X^* \rightarrow \overline{\mathbb{R}}$ , the conjugate of  $h$ , is defined on  $X^* \times X^{**}$ . Since there is a natural injection of  $X$  in to  $X^{**}$ , we define  $\mathcal{J}(h) : X \times X^* \rightarrow \overline{\mathbb{R}}$ ,

$$\begin{aligned} \mathcal{J}(h)(x, v) &= h^*(v, x), \\ &= \sup\{\langle (y, u), (v, x) \rangle - h(y, u); \mid (y, u) \in X \times X^*\}, \\ &= \sup\{\langle y, v \rangle + \langle x, u \rangle - h(y, u); \mid (y, u) \in X \times X^*\}. \end{aligned}$$

Fitzpatrick [4, Prop. 4.2] proved that if  $T : X \rightrightarrows X^*$  is maximal monotone,  $\mathcal{J}(\varphi_T)$  also belongs to  $\mathcal{H}(T)$ . In [2] this result was extended to any  $h \in \mathcal{H}(T)$ . Namely, if  $h \in \mathcal{H}(T)$ , then  $\mathcal{J}(h) \in \mathcal{H}(T)$ . Altogether, the result in [2] can be expressed as the implication

$$\begin{array}{l} T : X \rightrightarrows X^* \text{ maximal monotone} \\ h \in \mathcal{H}(T) \end{array} \Rightarrow \begin{array}{l} \forall (x, v) \in X \times X^*, \\ h(x, v) \geq \langle x, v \rangle, \quad h^*(v, x) \geq \langle x, v \rangle. \end{array}$$

Our aim is to prove the converse of this implication in a *reflexive* Banach space. Namely,

$$\begin{array}{l} h : X \times X^* \rightarrow \overline{\mathbb{R}} \text{ convex, lsc,} \\ \forall (x, v) \in X \times X^*, \\ h(x, v) \geq \langle x, v \rangle, \quad h^*(v, x) \geq \langle x, v \rangle. \end{array} \Rightarrow \begin{array}{l} \exists! T : X \rightrightarrows X^* \text{ maximal monotone} \\ h \in \mathcal{H}(T) \end{array}$$

The paper is organized as follows. In Section 2 we state some necessary previous results. The last section contains the formal statement and the proof of the implication above (see Section 3, Theorem 3.1).

## 2 Theoretical Preliminaries

**sec:te**

We include in this section theoretical results which are necessary for the proof of Theorem 3.1.

We assume from now on that  $X$  is a reflexive (real) Banach space. Asplund [1] has shown that, in this case, there exists an equivalent norm on  $X$

which is everywhere Gâteaux differentiable except at the origin and whose polar norm on  $X^*$  is everywhere Gâteaux differentiable except at the origin. For simplifying the notation, we assume that the given norm on  $X$  already has these special properties. We use the same notation  $\|\cdot\|$  for this norm on  $X$  and its associated norm on the dual  $X^*$ . Denote by  $J$  the Gâteaux gradient of the function  $g(x) = (1/2)\|x\|^2$ . Thus,  $J$  is the duality mapping, which assigns to each  $x \in X$  the unique  $J(x) \in X^*$  such that

$$\langle x, J(x) \rangle = \|x\|^2 = \|J(x)\|^2. \quad (2) \quad \boxed{\text{eq:rock.2.1}}$$

The inverse of this duality mapping will be denoted by  $J_*$ , which is the subgradient of the function  $g^*(v) = (1/2)\|v\|^2$ .

The following result was proved in [7, Section 2], where it appears as a corollary.

**pr:rock** **Proposition 2.1** *Let  $X$  be reflexive, and let  $J$  be the duality map defined above. Let  $T : X \rightrightarrows X^*$  be a monotone operator. In order that  $T$  be maximal monotone, it is necessary and sufficient that  $(T + J) : X \rightrightarrows X^*$  be onto.*

Using the fact that  $\langle z, u \rangle \geq -\|z\| \|u\|$  for all  $z \in X, u \in X^*$ , one can easily obtain the proposition below.

**pr:el** **Proposition 2.2** *Let  $X$  be reflexive, and let  $\|\cdot\|, J$  be the norm and duality map defined above, respectively. Take  $z \in X, u \in X^*$ . Then*

$$\|z\|^2 + \|u\|^2 + 2\langle z, u \rangle \geq 0,$$

*with equality if and only if  $u = -J(z)$  (or equivalently  $z = -J_*(u)$ ).*

Finally, we will also need the following result.

**th:jh** **Theorem 2.1 ([2, Theor. 5.3])** *Let  $T : X \rightrightarrows X^*$  be maximal monotone. Then, the operator  $\mathcal{J}$  maps  $\mathcal{H}(T)$  into itself.*

### 3 Main Result

**sec:main**

Now we state formally and prove the main result.

**th:main**

**Theorem 3.1** *Let  $X$  be reflexive and  $h : X \times X^* \rightarrow \overline{\mathbb{R}}$  be a convex lower semicontinuous function. Suppose that*

$$\forall (x, v) \in X \times X^*, \quad h(x, v) \geq \langle x, v \rangle, \quad h^*(v, x) \geq \langle x, v \rangle. \quad (3) \quad \text{eq:th.main.hyp}$$

*Define*

$$T = \{(x, v) \in X \times X^* \mid h(x, v) = \langle x, v \rangle\}. \quad (4) \quad \text{eq:th.main.defT}$$

*Then  $T$  is maximal monotone and  $h, \mathcal{J}(h) \in \mathcal{H}(T)$ .*

*Proof.* First we claim that  $T$  is monotone. Indeed, take  $v^1 \in T(x^1)$ ,  $v^2 \in T(x^2)$ , then

$$\langle x^1, v^1 \rangle = h(x^1, v^1), \quad \langle x^2, v^2 \rangle = h(x^2, v^2). \quad (5) \quad \text{eq:tc1}$$

The convexity of  $h$  together with (3) gives

$$\begin{aligned} (1/2)(h(x^1, v^1) + h(x^2, v^2)) &\geq h((1/2)(x^1 + x^2), (1/2)(v^1 + v^2)) \\ &\geq \langle (1/2)(x^1 + x^2), (1/2)(v^1 + v^2) \rangle \end{aligned} \quad (6) \quad \text{eq:tc2}$$

Combining this with (5) we obtain

$$(1/2) \left( \langle x^1, v^1 \rangle + \langle x^2, v^2 \rangle \right) \geq (1/4) \langle x^1 + x^2, v^1 + v^2 \rangle,$$

which is equivalent to  $\langle x^1 - x^2, v^1 - v^2 \rangle \geq 0$ .

Now we claim that  $T + J : X \rightrightarrows X^*$  is onto. To prove this fact, take an arbitrary  $v_0 \in X^*$  and define  $\varphi : X \times X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ ,

$$\begin{aligned} \varphi(x, v) &:= (1/2) \left( \|x\|^2 + \|v - v_0\|^2 + 2\langle x, v - v_0 \rangle \right) + \left( h(x, v) - \langle x, v \rangle \right) \\ &= (1/2) \left( \|v - v_0\|^2 + \|x\|^2 \right) - \langle v_0, x \rangle + h(x, v), \end{aligned} \quad (7) \quad \text{eq:defphi}$$

where  $\|\cdot\|, J$  are the norm and duality map defined above, respectively. By the first expression for  $\varphi$ , assumptions (3)-(4) and Proposition 2.2, we have  $\varphi \geq 0$ , with equality only if  $v - v_0 = -J(x)$  and  $v \in T(x)$ . This implies  $v_0 \in (T + J)(x)$ . The second expression of  $\varphi$  shows that this function is the sum of a differentiable convex function plus a lower semicontinuous convex function. By [8, Th. 3] or [3, p. 62] the subgradient of this sum is the

sum of the subgradients. Using also the equalities  $J(\cdot) = \partial(1/2\|\cdot\|^2)$  and  $J_*(\cdot) = \partial(1/2\|\cdot\|^2)$  we obtain

$$\begin{aligned}\partial\varphi(x, v) &= (\partial_X\varphi(x, v), \partial_{X^*}\varphi(x, v)) + \partial h(x, v) \\ &= (J(x) - v_0, J_*(v - v_0)) + \partial h(x, v).\end{aligned}$$

Since  $X \times X^*$  is reflexive and  $\varphi$  is lower semicontinuous and strongly convex, it attains a minimum at some  $(x, v) \in X \times X^*$ . Hence, for such  $(x, v)$

$$0 \in (J(x) - v_0, J_*(v - v_0)) + \partial h(x, v).$$

To simplify the manipulations, define

$$\begin{aligned}r &= J_*(v - v_0) + x, \\ \rho &= v - v_0 + J(x).\end{aligned}\tag{8} \quad \boxed{\text{eq:drr}}$$

With this notation, the last inclusion becomes

$$(v - \rho, x - r) \in \partial h(x, v).$$

Hence, by Fenchel-Young we have that

$$h(x, v) + h^*(v - \rho, x - r) = \langle x, v - \rho \rangle + \langle x - r, v \rangle.\tag{9} \quad \boxed{\text{eq:fy}}$$

Define now

$$C := \left( \langle x, v \rangle - h(x, v) \right) + \left( \langle v - \rho, x - r \rangle - h^*(v - \rho, x - r) \right).\tag{10} \quad \boxed{\text{eq:dc}}$$

Assumption (3) yields  $C \leq 0$ . Using now (9) we obtain

$$C = \langle r, \rho \rangle.$$

Applying (8) and the properties of  $J, J_*$  we obtain, after some manipulations

$$\begin{aligned}C &= (1/2)\left(\|x\|^2 + \|v - v_0\|^2 + 2\langle x, v - v_0 \rangle\right) \\ &\quad + (1/2)\left(\|J(x)\|^2 + \|J_*(v - v_0)\|^2 + 2\langle J_*(v - v_0), J(x) \rangle\right).\end{aligned}\tag{11} \quad \boxed{\text{eq:c2}}$$

By the above equation and Proposition 2.2,  $C \geq 0$ . Therefore,  $C = 0$ . Using this fact and again (11) and Proposition 2.2, we conclude that  $v - v_0 = -J(x)$ , that is

$$v + J(x) = v_0.\tag{12} \quad \boxed{\text{eq:eq}}$$

On the other hand, using (10), (3) and the equality  $C = 0$  we conclude that  $h(x, v) = \langle v, x \rangle$ , which yields

$$v \in T(x).$$

Therefore,  $v_0 \in (T + J)(x)$ . Since  $v_0$  is arbitrary,  $T + J$  is onto.

We have thus proved that  $T$  is monotone and  $T + J$  is onto, hence by Proposition 2.1,  $T$  is maximal monotone. It remains to prove that  $h$  and  $\mathcal{J}(h) \in \mathcal{H}(T)$ . In order to do this, we use (3) and the definition of  $\mathcal{H}$ , for concluding that  $h \in \mathcal{H}(T)$ . The inclusion  $\mathcal{J}(h) \in \mathcal{H}(T)$  now follows from Theorem 2.1.  $\square$

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